Two-proton correlation function: a gentle introduction

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Abstract. The recent COSY-11 collaboration measurement of the two-proton correlation function in the $pp \to pp\eta$ reaction, reported at this meeting [1], arouse some interest in a simple theoretical description of the correlation function. In these notes we present a pedagogical introduction to the practical methods that can be used for calculating the correlation function.

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We are going here to deal with low-energy phenomena and in order to avoid unnecessary complications our approach will be non-relativistic. We wish to develop a practical scheme for calculating the two-particle correlation function $C(k)$, with k denoting their relative momentum, choosing as the departure point the familiar formula

$$C(k) = \int D(r)\, |\Phi_{\boldsymbol{k}}(\boldsymbol{r})|^2 \, \mathrm{d}^3 r, \tag{1}$$

(referred to in the literature as the Koonin-Pratt model [2]), expressing $C(k)$ as an overlap of two distributions. The first distribution, $D(r)$, is the *effective* source density function that is usually for convenience assumed to be a Gaussian

$$D(r) = (4\pi d^2)^{-3/2}\, \mathrm{e}^{-(r/2d)^2} \tag{2}$$

with a single parameter d reflecting the size of the source. The second factor entering the overlap integral (1), is a probability density involving the square of the wave function $\Phi_{\boldsymbol{k}}(\boldsymbol{r})$ describing two-particle system in the continuum.

Actually, formula (1) is only a static approximation to the correlation function derived under the assumption that the final-state interaction between the two detected particles dominates, while all other interactions are negligible. Furthermore, it is assumed that the correlation functions are determined by the two-body densities of states (corrected for their mutual interactions) and that the single particle phase space distribution function of the emitted particle varies slowly with momentum. Admittedly, the question concerning the validity of these assumptions is far from settled but since they result in a manageable calculational scheme it is worthwhile to examine in some detail its consequences.

Adopting hereafter units in which $\hbar = c = 1$, for non-interacting particles $\Phi_{\boldsymbol{k}}(\boldsymbol{r})$ takes the form of a plane wave

$$\Phi_{\boldsymbol{k}}(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

and in this case the correlation function is equal to unity, which means no correlation. However, when the particles are non-interacting identical bosons(fermions), the plane

wave needs to be properly symmetrized(antisymmetrized), viz.

$$\Phi_{\boldsymbol{q}}^{\pm}(\boldsymbol{r}) = \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \pm \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{2}}$$

and the interference term yields a non-vanishing contribution to the correlation function in excess of unity. For a Gaussian source, direct integration gives

$$C_{\pm}(k) = 1 \pm \mathrm{e}^{-4d^2k^2} \tag{3}$$

and in the symmetric case $C_+(k)$ peaks at threshold with $C_+(0) = 2$ going eventually to unity at large k. By contrast, in the antisymmetric case $C_-(k)$ vanishes at threshold reaching unity from below for large k. The rate at which the correlation function approaches unity is controlled by d, the only parameter in the game. Actually, $C_{\pm}(k)$ could be viewed as a function depending on two parameters k and d but for free propagation, owing to dimensional scaling, the correlation function can only depend upon their product dk bearing a universal character. For interacting particles, as soon as additional parameters become available, further dimensionless quantities can be formed and dk is no longer the only possible combination.

When the two-particle system has additional degrees of freedom the even and odd components of the wave function may both enter the correlation function. This happens in the two-proton case where the isospin part of the wave function is necessarily symmetric and therefore Pauli principle admits in the spin-singlet states only even ℓ whereas in spin-triplet states, respectively, odd ℓ. In the simplest case of two non-interacting protons the resulting spin weights are purely statistical, and we get

$$C_{pp}(k) = \frac{1}{4}C_+(k) + \frac{3}{4}C_-(k) = 1 - \frac{1}{2}\mathrm{e}^{-4d^2k^2} \tag{4}$$

with the intercept $C_{pp}(0) = \frac{1}{2}$.

Apart from correlations associated with permutation symmetry there would be also dynamical correlations induced by the two-particle interaction. To get some insight into the nature of dynamical correlations let us consider the case of two different particles whose propagation is not free but distorted by their mutual interaction. For simplicity we assume that the interaction may be represented by a spherically symmetric potential $V(r)$. The wave function may be expanded in partial waves

$$\Phi_{\boldsymbol{k}}(\boldsymbol{r}) = \sum_{\ell=0}^{\infty}(2\ell+1)\mathrm{i}^{\ell}\, u_{\ell}(k,r)/(kr)\, P_{\ell}(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}}) \tag{5}$$

where $P_\ell(z)$ denotes Legendre polynomial and the function $u_\ell(k,r)$ is the solution of the appropriate radial Schrödinger equation and for $r \to \infty$ satisfies the outgoing wave boundary condition

$$u_{\ell}(k,r) \sim \sin(kr - \tfrac{1}{2}\ell\pi) + k f_{\ell}(k)\, \exp\left[\mathrm{i}(kr - \tfrac{1}{2}\ell\pi)\right] \tag{6}$$

where $f_\ell(k)$ is the partial wave scattering amplitude. Inserting (5) in (1), after trivial angular integration, the correlation function takes the form of a series

$$C(k) = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\int_0^\infty D(r)\,|u_\ell(k,r)|^2\,\mathrm{d}r \tag{7}$$

whose convergence rate depends upon the case considered. For a short ranged potential the functions $|u_\ell(k,r)|^2$ are pushed outward by the centrifugal barrier and their overlap with the density rapidly decreases with increasing ℓ and in consequence the series (7) also rapidly converges. The same reasoning repeated for the symmetric wave functions, yields

$$C_+(k) = \frac{8\pi}{k^2}\sum_{\ell=even}(2\ell+1)\int_0^\infty D(r)\,|u_\ell(k,r)|^2\,\mathrm{d}r \tag{8}$$

and for $C_-(k)$ analogous formula holds but the summation would be over odd ℓ.

As an important example we consider the Coulomb interaction $V_c(r) = \alpha/r$ where α is the fine structure constant times a product of the charge numbers. The radial functions are well known

$$u_\ell(k,r) = F_\ell(\eta,\rho) \tag{9}$$

with $\rho = kr$, $\quad \eta = \alpha\mu/k$ where μ is the reduced mass and the regular Coulomb functions $F_\ell(\eta,\rho)$ are defined in Abramowitz and Stegun [3] (for numerical methods of computing them cf. [4] and references therein). If the two protons experienced only Coulomb interaction, the pp correlation function respecting Pauli principle would be

$$\begin{aligned} C_{pp}(k) = \frac{2\pi}{k^2}\Bigg\{&\sum_{\ell=even}(2\ell+1)\int_0^\infty D(r)\,F_\ell(\eta,kr)^2\,\mathrm{d}r\,+ \\ &+3\sum_{\ell=odd}(2\ell+1)\int_0^\infty D(r)\,F_\ell(\eta,kr)^2\,\mathrm{d}r\Bigg\} \end{aligned} \tag{10}$$

where the integrals require numerical treatment.

The last step is the inclusion of strong interaction which, in general, leads to complications stemming from the fact that the nuclear forces have non-central components like the tensor interaction or spin-orbit force, and the orbital momentum is not a good quantum number. However, for low energies ($k < 100\,MeV/c$, say) the dominant contribution comes from the s-wave and the interaction in higher partial waves can be neglected. In this case ℓ is still a good quantum number and including both, the Coulomb and the nuclear s-wave interaction, the pp correlation function respecting Pauli principle, reads

$$\begin{aligned} C_{pp}(k) = \frac{2\pi}{k^2}\Bigg\{\int_0^\infty D(r)\,|u_0(k,r)|^2\,\mathrm{d}r + \sum_{\ell=even>0}(2\ell+1)\int_0^\infty D(r)\,F_\ell(\eta,kr)^2\,\mathrm{d}r\Bigg\} + \\ + \frac{6\pi}{k^2}\sum_{\ell=odd}(2\ell+1)\int_0^\infty D(r)\,F_\ell(\eta,kr)^2\,\mathrm{d}r \end{aligned} \tag{11}$$

where $u_0(k,r)$ is the solution of the wave equation for $\ell = 0$ involving both, the Coulomb and the strong interaction potential. The above formula constitutes the basis for all our computations.

To get a feeling how (11) works in practice it is instructive to provide some example. Perhaps the simplest simulation of the nuclear force delivers the delta-shell potential model when $V(r)$ is taken in the form

$$2\mu V(r) = -(s/R)\,\delta(r-R) \tag{12}$$

where s is a dimensionless strength parameter and R is the range parameter, assuming that the potential (12) is operative in s-wave only. The wave function $u_0(k,r)$ may be obtained by solving the appropriate Schrödinger equation but here we shall use for that purpose the equivalent Lippmann-Schwinger equation which incorporates the correct asymptotic boundary condition. More explicitly, we are presented with the integral equation

$$u_0(k,r) = F_0(\eta,kr) + 2\mu \int_0^\infty g_0^+(r,r')\,V(r')\,u_0(k,r')\,\mathrm{d}r' \tag{13}$$

where $g_0^+(r,r')$ is the outgoing wave Coulomb Green's function

$$g_0^+(r,r') = -(1/k)\,F_0(\eta,kr_<)\,[G_0(\eta,kr_>) + \mathrm{i}\,F_0(\eta,kr_>)] \tag{14}$$

with $G_\ell(\eta,kr)$ denoting the irregular Coulomb wave function (cf. [3], [4]) and the other symbols are: $r_< = \min(r,r')$; $r_> = \max(r,r')$. Note, that acting with the operator $\mathrm{d}^2/\mathrm{d}r^2 + k^2 - 2\mu\alpha/r$ on both sides of (13), one immediately recovers the Schrödinger equation. For the potential (12) it is a trivial matter to obtain the solution of (13) and the latter is given in an analytic form

$$u_0(k,r) = \begin{cases} A(k)\,F_0(\eta,kr) & \text{for} \quad r \le R \\ F_0(\eta,kr) + f_0(k)\,k[G_0(\eta,kr) + \mathrm{i}\,F_0(\eta,kr)] & \text{for} \quad r \ge R \end{cases} \tag{15}$$

where the amplitude $A(k)$ is

$$A(k) = \frac{1}{1 - (s/kR)\,F_0(\eta,kR)\,[G_0(\eta,kR) + \mathrm{i}\,F_0(\eta,kR)]} \tag{16}$$

and the second constant $f_0(k)$ which is the appropriate pp scattering amplitude, is given as

$$f_0(k) = (s/k^2R)F_0(\eta,kR)^2\,A(k). \tag{17}$$

Feeding (11) with $u_0(k,r)$ given in (15), and performing (numerically) the integrations we obtain the correlation function. For "realistic" nuclear potentials the procedure would be similar, the only difference being that $u_0(k,r)$ must be generated numerically from (13).

In our computations we have tried a variety of NN potentials (Reid68, Reid93, Nijmegen93 and Argonne AV18) but the corresponding correlation functions were in all cases nearly indistinguishable. On the other hand, the delta-shell potential resulted in

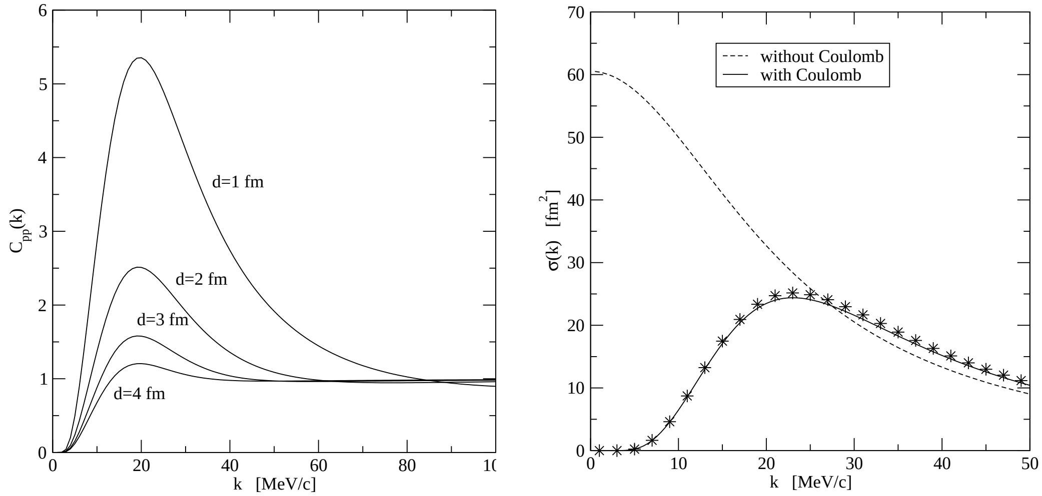

FIGURE 1. (left) Two-proton correlation function versus k for different source radii calculated from the Reid soft core potential. (right) Partial wave ($\ell = 0$) pp cross-section calculated from (19). The stars show the cross-section computed from the toy model formula (17) for $s = 0.906$ and $R = 1.84$ fm.

markedly different shape of the correlation function. The plots of the correlation function calculated for Reid soft core potential are presented in Fig.1(left) for different radii. In qualitative terms the curves are similar: all show a depression close to threshold and a prominent peak at about 20 MeV/c. The same behavior has been observed experimentally [1]. In our model the position of the peak does not seem to be dependent upon d whilst its height is very sensitive to the source radius. The depression close to threshold results from a combined effect of Fermi statistics and Coulomb repulsion which both try to keep the two protons apart causing the correlation to be negative at small k. The peak at 20 MeV/c has a dynamic origin, manifesting strong nuclear attraction experienced by the protons. The height of the peak decreases rapidly for increasing d assuming the largest possible value for $d = 0$. In the latter case the source shrinks to a point and the density (2) can be replaced by a delta function $D(r) = \delta(\boldsymbol{r})$. The correlation function for a point-like source can be immediately obtained, and reads

$$\lim_{d\to 0} C_{pp}(k) = \frac{1}{4} \lim_{r\to 0} \frac{|u_0(k,r)|^2}{(kr)^2} = \frac{1}{4k^2}\ |u_0'(k,0)|^2, \tag{18}$$

where prime denotes derivative with respect to r. The expression on the right hand side of (18) is nothing else but the enhancement factor associated with pp final-state interaction, used e.g. to approximate the $pp \to pp\eta$ cross section, and which is also known to have a prominent peak at 20 MeV/c.

To understand better the origin of the peak at 20 MeV/c let us consider first a simplified situation with Coulomb interaction switched off. Close to threshold the NN $\ell = 0$ scattering amplitude may be expressed in terms of the scattering length a as $f_0(k) = (-1/a - \mathrm{i}k)^{-1}$ and in the complex k-plane this amplitude has a pole at $k_p = \mathrm{i}/a$. For $a > 0$ the pole is on the physical sheet ($\Im k_p > 0$) and we have a bound state. This is the deuteron case. However, when $a < 0$, the pole is on the non-physical sheet ($\Im k_p < 0$) and we are left with a virtual state. For the pp pair this would be the particle unstable

^2He state. In either case the NN cross section shows a pronounced peak at threshold as seen in Fig.1(right) (dashed curve). Putting back the Coulomb interaction and including the effective range (r_0) term, a model independent expression for the low-energy pp scattering amplitude, takes the form

$$f_0(k) \approx \frac{C_0^2(\eta)}{-1/a + \frac{1}{2}r_0k^2 - 2k\eta h(\eta) - \mathrm{i}kC_0^2(\eta)} \tag{19}$$

where $C_0^2(\eta) = 2\pi\eta/[\exp(2\pi\eta) - 1])$ is the Gamow factor, $h(\eta) = \Re\psi(1+\mathrm{i}\eta) - \log|\eta|$ and ψ denotes the digamma function [3]. The corresponding partial wave cross-section, i.e. $|f_0(k)|^2$ for a=-7.78 fm and r_0=2.72 fm is presented in Fig.1(right) by the continuous curve and a similar result would be obtained from the toy model formula (17) (depicted by stars in Fig.1(right)). Owing to the Coulomb corrections, the peak in the cross section is shifted off-threshold by about 20 Mev/c and its height is depressed to one third of its original value. This is now the very same peak that appears in the correlation function and can be regarded as an artifact of the ^2He state. It is *not* a resonance though because the real part of the denominator in (19) equal $-1/a + \frac{1}{2}r_0k^2 - 2k\eta h(\eta)$ does not vanish, showing instead only a minimum whose position is identical with the position of the peak in Fig.1(right). Actually, given the low-energy pp scattering parameters, the position of the peak could have been predicted by equating to zero the derivative of the real part of the denominator in (19) and by solving the resulting equation for k.

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